# STIRLING PERMUTATIONS, CYCLE STRUCTURES OF PERMUTATIONS AND PERFECT MATCHINGS

SHI-MEI MA AND YEONG-NAN YEH

ABSTRACT. In this paper we provide a unified combinatorial approach to establish a connection between Stirling permutations, cycle structures of permutations and perfect matchings. The main tool of our investigations is MY-sequences. In particular, we discover that the Eulerian polynomials have a simple combinatorial interpretation in terms of some statistics on MY-sequences.

*Keywords:* Stirling permutations; Eulerian polynomials; Perfect matchings; MY-sequences

## 1. INTRODUCTION

Stirling permutations were defined by Gessel and Stanley [6]. Let $j^2 := \{j, j\}$ for $j \geq 1$. A *Stirling permutation* of order $n$ is a permutation of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that every element between the two occurrences of $i$ are greater than $i$ for each $i \in [n]$, where $[n] = \{1, 2, \ldots, n\}$. Denote by $\mathcal{Q}_n$ the set of Stirling permutations of order $n$. For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in \mathcal{Q}_n$, an occurrence of an *ascent* (resp. *a plateau*) is an index $i$ such that $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i = \sigma_{i+1}$), Recently, there is a large literature devoted to Stirling permutations and their generalizations. The reader is referred to [1, 8, 9, 13, 14] for recent progress on the study of statistics on Stirling permutations.

In this paper, we always assume that Stirling permutations are prepended by 0. That is, we identify an $n$-Striling permutation $\sigma_1 \sigma_2 \cdots \sigma_{2n}$ with the word $\sigma_0 \sigma_1 \sigma_2 \cdots \sigma_{2n}$, where $\sigma_0 = 0$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in \mathcal{Q}_n$. We say that an index $i \in [2n-1]$ is an *ascent plateau* if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$ (see [13]). Let ap $(\sigma)$ be the number of the ascent plateaus of $\sigma$. For example, ap $(\mathbf{221133}) = 2$.

We define

$$N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)}. \tag{1}$$

Analyzing the placement of 2 copies of $(n + 1)$, it is easy to deduce that the polynomials $N_n(x)$ satisfy the recurrence relation

$$N_{n+1}(x) = (2n + 1)x N_n(x) + 2x(1 - x)N_n'(x)$$

with the initial value $N_0(x) = 1$. The exponential generating function for $N_n(x)$ is given as follows (see [11, Section 5]):

$$N(x, z) = \sum_{n \geq 0} N_n(x) \frac{z^n}{n!} = \sqrt{\frac{1 - x}{1 - x e^{2z(1-x)}}}. \tag{2}$$

The first few of $N_n(x)$ are

$$N_1(x) = x, N_2(x) = 2x + x^2, N_3(x) = 4x + 10x^2 + x^3, N_4(x) = 8x + 60x^2 + 36x^3 + x^4.$$

Let $\mathfrak{S}_n$ denote the permutation group on the set $[n]$ and $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *excedance* in $\pi$ is an index $i$ such that $\pi(i) > i$. Let $\mathrm{exc}\,(\pi)$ denote the number of excedances in $\pi$. The classical Eulerian polynomials $A_n(x)$ are defined by

$$A_0(x) = 1, \quad A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} \quad \text{for } n \geq 1,$$

and have been extensively investigated (see [4, 5, 7] for instance). In [5], Foata and Schützenberger introduced a $q$-analog of the Eulerian polynomials defined by

$$A_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}.$$

where $\mathrm{cyc}\,(\pi)$ is the number of cycles in $\pi$. Brenti [2, 3] further studied $q$-Eulerian polynomials and established the link with $q$-symmetric functions arising from plethysm. In particular, Brenti [3, Proposition 7.3] obtained the exponential generating function for $A_n(x; q)$:

$$1 + \sum_{n \geq 1} A_n(x; q)\frac{z^n}{n!} = \left(\frac{1 - x}{e^{z(x-1)} - x}\right)^q.$$

For $k \geq 1$, the *$1/k$-Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n \geq 0} A_n^{(k)}(x)\frac{z^n}{n!} = \left(\frac{1 - x}{e^{kz(x-1)} - x}\right)^{\frac{1}{k}}. \tag{3}$$

Let $e = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n$. Let $I_{n,k} = \{e | 0 \leq e_i \leq (i-1)k\}$, which known as the set of $n$-dimensional *$k$-inversion sequences* (see [15]). The number of *ascents* of $e$ is defined by

$$\mathrm{asc}\,(e) = \#\left\{i : 1 \leq i \leq n - 1 \Big| \frac{e_i}{(i-1)k + 1} < \frac{e_{i+1}}{ik + 1}\right\}.$$

Recently, Savage and Viswanathan [16] discovered that

$$A_n^{(k)}(x) = \sum_{e \in I_{n,k}} x^{\mathrm{asc}\,(e)} = k^n A_n(x; 1/k).$$

From (2) and (3), we get

$$A_n^{(2)}(x) = x^n N_n\left(\frac{1}{x}\right). \tag{4}$$

Hence

$$N_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{n-\mathrm{exc}\,(\pi)} 2^{n-\mathrm{cyc}\,(\pi)}. \tag{5}$$

Let $\widehat{N}_n(x) = x^n N_n\left(\frac{1}{x}\right)$. It follows from (3) and (4) that

$$2^n A_n(x) = \sum_{k=0}^{n} \binom{n}{k} \widehat{N}_k(x)\widehat{N}_{n-k}(x).$$

A *perfect matching* of $[2n]$ is a set partition of $[2n]$ with blocks (disjoint nonempty subsets) of size exactly 2. Let $\mathcal{M}_{2n}$ be the set of matchings of $[2n]$, and let $\mathrm{M} \in \mathcal{M}_{2n}$. The *standard form* of M is a list of blocks $(i_1, j_1)/(i_2, j_2)/\ldots/(i_n, j_n)$ such that $i_r < j_r$ for all $1 \leq r \leq n$

and $1 = i_1 < i_2 < \cdots < i_n$. Throughout this paper we always write M in standard form. It is well known that M can be regarded as a fixed-point-free involution on $[2n]$ and also as a Brauer diagram on $[2n]$ (see [10] for instance). Let so (M) be the number of blocks of M with odd smaller entries. It is well known that

$$N_n(x) = \sum_{M \in \mathcal{M}_{2n}} x^{so\,(M)}, \tag{6}$$

which has been studied in [11, 12]. In particular, from [11, Theorem 9], we have

$$2^n x A_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}(x).$$

Combining (1), (5) and (6), we get

$$\sum_{\sigma \in \mathcal{Q}_n} x^{ap\,(\sigma)} = \sum_{\pi \in \mathfrak{S}_n} x^{n - exc\,(\pi)} 2^{n - cyc\,(\pi)} = \sum_{M \in \mathcal{M}_{2n}} x^{so\,(M)}. \tag{7}$$

It is natural to consider the following question: Is there existing a unified combinatorial approach to prove (7)? The main object of this paper is to provide a solution to this problem. The paper is organized as follows. In Section 2, we present the main results, and recall some definitions that will be used throughout the rest of this work. In other Sections, we give combinatorial proofs of these main results.

## 2. Definitions and main results

Let $N_n(x) = \sum_{k=1}^{n} N(n,k) x^k$. Hence $N(n,k)$ is the number of perfect matchings in $\mathcal{M}_{2n}$ with the restriction that only $k$ matching pairs have odd smaller entries. We partition the blocks of M into four subsets:

$$OO(M) = \{(a,b) \in M \mid a \text{ and } b \text{ are both odd}\},$$
$$OE(M) = \{(a,b) \in M \mid a \text{ is odd and } b \text{ is even}\},$$
$$EO(M) = \{(a,b) \in M \mid a \text{ is even and } b \text{ is odd}\},$$
$$EE(M) = \{(a,b) \in M \mid a \text{ and } b \text{ are both even}\}.$$

Let $oo\,(M) = \#OO(M), oe\,(M) = \#OE(M), eo\,(M) = \#EO(M), ee\,(M) = \#EE(M)$. It is evident that $so\,(M) = oo\,(M) + oe\,(M)$. Here we present a further characterization of $so\,(M)$.

**Proposition 1.** *We have* $so\,(M) = oe\,(M) + ee\,(M)$, *and the numbers* $N(n,k)$ *satisfy the recurrence relation*

$$N(n+1,k) = 2k N(n,k) + (2n - 2k + 3) N(n, k-1). \tag{8}$$

*for* $n, k \geq 1$, *where* $N(1,1) = 1$ *and* $N(1,k) = 0$ *for* $k \geq 2$ *or* $k \leq 0$.

*Proof.* Given a $M \in \mathcal{M}_{2n}$. Note that

$$2oo\,(M) + \#oe\,(M) + \#eo\,(M) = n = 2\#ee\,(M) + \#oe\,(M) + \#eo\,(M).$$

Then $\#oo\,(M) = \#ee\,(M)$. Hence $so\,(M) = oe\,(M) + ee\,(M)$. Therefore, we have

$$N(n,k) = \#\{M \in \mathcal{M}_{2n} : oe\,(M) + ee\,(M) = k\}. \tag{9}$$

We now prove (8). Let $(a, b)$ be a given subset of M. Let $\varphi$ be the construction of $M' \in \mathcal{M}_{2n+2}$ that replacing $(a, b)$ by $(a, 2n+1)/(b, 2n+2)$ or $(a, 2n+2)/(b, 2n+1)$. We distinguish two cases.

$(c_1)$ If $(a, b) \in \text{OE}$ or $(a, b) \in \text{EE}$, then the construction $\varphi$ does not increasing the number odd smallers. Combining (9), this accounts for $2kN(n, k)$ possibilities.

$(c_2)$ If $(a, b) \in \text{OO}$ or $(a, b) \in \text{EO}$, then the construction $\varphi$ does form a new odd smaller. Moreover, we can also append $(2n + 1, 2n + 2)$ to M. This gives $(2n - 2(k - 1) + 1)N(n, k - 1) = (2n - 2k + 3)N(n, k - 1)$ possibilities.

$\square$

For $k \geq 1$ and $\ell \geq 0$, we define

$$P_k = \{1, 2, 3, \ldots, 2k\} \text{ and } N_\ell = \{-1, -2, -3, \ldots, -2\ell, \star\}.$$

Let $Y_n = (y_1, y_2, \ldots, y_n)$, where $y_i \in P_k \bigcup N_\ell$ for $1 \leq i \leq n$. In particular, $P_1 \bigcup N_0 = \{1, 2, \star\}$, $P_1 \bigcup N_1 = \{1, 2, -1, -2, \star\}$ and $P_2 \bigcup N_0 = \{1, 2, 3, 4, \star\}$. Let $\text{pos}(Y_n)$ (resp. $\text{neg}(Y_n)$) be the number of positive (resp. negative) entries of $Y_n$. Denote by $\text{star}(Y_n)$ the number of $\star$ of $Y_n$. Careful consideration of Proposition 1 yields the following definition.

**Definition 2.** *We call the sequence $Y_n$ a MY-sequence of length $n$ if $y_1 = \star$ and $y_k \in P_{1+s_k} \bigcup N_{t_k}$ for $2 \leq k \leq n$, where $s_k = \text{neg}(Y_{k-1}) + \text{star}(Y_{k-1}) - 1$ and $t_k = \text{pos}(Y_{k-1})$ for $k \geq 2$.*

Note that $s_2 = t_2 = 0$. Hence $y_2 \in P_1 \bigcup N_0$. For example, $(\star, 1, -1, 2, \star)$ is a MY-sequence, while $(\star, 1, -1, -4, 2)$ is not since $y_4 < -2$. Denote by $\mathcal{Y}_n$ the set of MY-sequences $Y_n$. Note that $1 + s_k + t_k = k - 1$ for $k \geq 2$. Therefore, we have

$$\#\mathcal{Y}_n = (2(1 + s_n) + (1 + 2t_n))\#\mathcal{Y}_{n-1} = (2n - 1)\#\mathcal{Y}_{n-1} = (2n - 1)!!.$$

We can now present the first main result of this paper.

**Theorem 3.** *For $n \geq 1$, we have*

$$\sum_{M \in \mathcal{M}_{2n}} x^{\text{oe}(M) + \text{ee}(M)} = \sum_{Y_n \in \mathcal{Y}_n} x^{\text{neg}(Y_n) + \text{star}(Y_n)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)}. \tag{10}$$

In this paper we always write $\pi \in \mathfrak{S}_n$ by its standard cycle decomposition, in which each cycle is written with its smallest entry first and the cycles are written in ascending order of their smallest entry.

**Definition 4.** *Let $(c_1, c_2, \ldots, c_i)$ be one cycle of $\pi$. We say that $c_j$ is a cycle descent if $c_j > c_{j+1}$ for $1 \leq j < i$.*

Let $\text{cdes}(\pi)$ be the number of cycle descents of $\pi$. For example, for $\pi = (1, 3, \mathbf{4}, 2)(5, 7)(6)$, we have $\text{cdes}(\pi) = 1$. For $\pi \in \mathfrak{S}_n$, it is clearly that $\text{exc}(\pi) + \text{cyc}(\pi) + \text{cdes}(\pi) = n$. Therefore, it follows from (5) that

$$N_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{cyc}(\pi) + \text{cdes}(\pi)} 2^{n - \text{cyc}(\pi)}.$$

Now we present the second main result of this paper.

**Theorem 5.** *For $n \geq 1$, we have*

$$\sum_{Y_n \in \mathcal{Y}_n} x^{\mathrm{neg}\,(Y_n)} y^{\mathrm{star}\,(Y_n)} z^{\mathrm{pos}\,(Y_n)} = \sum_{\pi \in \mathfrak{S}_n} (2x)^{\mathrm{cdes}\,(\pi)} y^{\mathrm{cyc}\,(\pi)} (2z)^{\mathrm{exc}\,(\pi)}. \tag{11}$$

Note that $\mathrm{neg}\,(Y_n) + \mathrm{star}\,(Y_n) + \mathrm{pos}\,(Y_n) = n$. The following corollary is immediate.

**Corollary 6.** *For $n \geq 1$, we have*

$$A_n(x) = \sum_{Y_n \in \mathcal{Y}_n} \left(\frac{1}{2}\right)^{\mathrm{neg}\,(Y_n)+\mathrm{pos}\,(Y_n)} x^{\mathrm{pos}\,(Y_n)}.$$

*Equivalently,*

$$2^n A_n(x) = \sum_{Y_n \in \mathcal{Y}_n} 2^{\mathrm{star}\,(Y_n)} x^{\mathrm{pos}\,(Y_n)}.$$

In the following sections, we give bijective proofs of the main results. As a consequence, we get a desired proof of (7).

## 3. Proof of the left equality of (10)

In the following discussion, we always put any 2-elements block of M into exactly a parenthesis or a square bracket. For $(a, b) \in \mathrm{M}$, if $(a, b) \in \mathrm{OE}(\mathrm{M})$ or $(a, b) \in \mathrm{EE}(\mathrm{M})$, then we replace $(a, b)$ by $[a, b]$. Otherwise, the parentheses that contain $a$ and $b$ unchanged. For example, we replace $(1, 3)/(2, 4)/(5, 8)/(6, 7)$ by $(1, 3)/[2, 4]/[5, 8]/(6, 7)$. Set $m = 2(1 + s_k + t_k)$. Let $\mathcal{M}\,(P_{1+s_k} \bigcup N_{t_k})$ be the set of perfect matchings of $[m]$ with exactly $1 + s_k$ square brackets and $t_k$ parentheses.

Now we present a bracket-breaking algorithm (BB algorithm for short):

$(S_1)$ Note that $\mathcal{M}(P_1 \bigcup N_0) = [1, 2]$. We take $[1, 2]$ as the starting point, which corresponds to the first term of a MY-sequence. Since $P_1 \bigcup N_0 = \{1, 2, \star\}$, we can perform one of the following three operations:

$(c_1)$ Using the element 1 of $P_1 \bigcup N_0$ to break the square bracket $[1, 2]$ such that $[1, 2]$ is replaced by $(1, 3)/[2, 4]$.

$(c_2)$ Using the element 2 of $P_1 \bigcup N_0$ to break the square bracket $[1, 2]$ such that $[1, 2]$ is replaced by $[1, 4]/(2, 3)$.

$(c_3)$ As for $\star$, we append $[3, 4]$ right after $[1, 2]$.

$(S_2)$ For $k \geq 2$, given a $\mathrm{M} \in \mathcal{M}(P_{1+s_k} \bigcup N_{t_k})$. We use entries of $P_{1+s_k} \bigcup N_{t_k}$ to break some square brackets or parentheses of M. Assume that $c, d, e$ and $f$ are positive integers. For the entries of $P_{1+s_k}$, we distinguish two cases:

$(c_1)$ Using the element $2i - 1$ to break the $i$-th square bracket with the restriction that (i) if the $i$-th square bracket with elements $2c - 1$ and $2d$, then we replace $[2c-1, 2d]$ by $(2c - 1, m + 1)/[2d, m + 2]$; (ii) if the $i$-th square bracket with elements $2e$ and $2f$, then we replace $[2e, 2f]$ by $(2e, m + 1)/[2f, m + 2]$.

$(c_2)$ Using the element $2i$ to break the $i$-th square bracket with the restriction that (i) if the $i$-th square bracket with elements $2c - 1$ and $2d$, then we replace $[2c - 1, 2d]$ by $[2c - 1, m + 2]/(2d, m + 1)$; (ii) if the $i$-th bracket with elements $2e$ and $2f$, then we replace $[2e, 2f]$ by $[2e, m + 2]/(2f, m + 1)$.

For the entries of $N_{t_k}$, we distinguish three cases:

($c_1$) For $1 \leq i \leq t_k$, we use the element $-2i + 1$ to break the $i$-th parenthesis with the restriction that (i) if the $i$-th parenthesis with elements $2c - 1$ and $2d - 1$, then we replace $(2c-1, 2d-1)$ by $(2c-1, m+1)/[2d-1, m+2]$; (ii) if the $i$-th parenthesis with elements $2e$ and $2f - 1$, then we replace $(2e, 2f - 1)$ by $(2e, m + 1)/[2f - 1, m + 2]$.

($c_2$) For $1 \leq i \leq t_k$, we use the element $-2i$ to break the $i$-th parenthesis with the restriction that (i) if the $i$-th parenthesis with elements $2c - 1$ and $2d - 1$, then we replace $(2c - 1, 2d - 1)$ by $[2c - 1, m + 2]/(2d - 1, m + 1)$; (ii) if the $i$-th parenthesis with elements $2e$ and $2f-1$, then we replace $(2e, 2f-1)$ by $[2e, m+2]/[2f-1, m+1]$.

($c_3$) As for $\star$, we append $[m + 1, m + 2]$ right after M.

Given a MY-sequence $Y_n$ with $\mathrm{neg}\,(Y_n) + \mathrm{star}\,(Y_n) = i$. Repeat the BB algorithm $n$-times, we can get a unique perfect matching M of $[2n]$ with $\mathrm{oe}\,(M) + \mathrm{ee}\,(M) = i$. Conversely, given a perfect matching $M \in \mathcal{M}_{2n}$ with $\mathrm{oe}\,(M) + \mathrm{ee}\,(M) = i$. If we delete $2n - 1$ and $2n$, then we can find the $n$-th element of the corresponding MY-sequence. Along the same lines, we can get a unique MY-sequence $Y_n$ with $\mathrm{neg}\,(Y_n) + \mathrm{star}\,(Y_n) = i$. Thus the BB algorithm gives a bijective proof of the left equality of (10). In fact, using the BB algorithm, we give a bijective proof of the following result:

$$\sum_{M \in \mathcal{M}_{2n}} x^{\mathrm{oe}\,(M)+\mathrm{ee}\,(M)} y^{\mathrm{eo}\,(M)+\mathrm{oo}\,(M)} = \sum_{Y_n \in \mathcal{Y}_n} x^{\mathrm{neg}\,(Y_n)+\mathrm{star}\,(Y_n)} y^{\mathrm{pos}\,(Y_n)}.$$

**Example 7.** *Let $Y_5 = (\star, 1, -2, 4, -1)$ and let $M = [1, 6]/[2, 8]/(3, 9)/(4, 7)/[5, 10]$. The correspondence between $Y_5$ and M is built up as follows:*

$1 \to [1, 2] \Leftrightarrow (1, 3)/[2, 4]$;

$-2 \to (1, 3)/[2, 4] \Leftrightarrow [1, 6]/(3, 5)/[2, 4] = [1, 6]/[2, 4]/(3, 5)$;

$4 \to [1, 6]/[2, 4]/(3, 5) \Leftrightarrow [1, 6]/[2, 8]/(4, 7)/(3, 5) = [1, 6]/[2, 8]/(3, 5)/(4, 7)$;

$-1 \to [1, 6]/[2, 8]/(3, 5)/(4, 7) \Leftrightarrow [1, 6]/[2, 8]/(3, 9)/[5, 10]/(4, 7) = [1, 6]/[2, 8]/(3, 9)/(4, 7)/[5, 10]$.

## 4. Proof of the right equality of (10)

Set $r = 1 + s_k + t_k$. Denote by $\mathcal{Q}\,(P_{1+s_k} \bigcup N_{t_k})$ the set of Stirling permutations of order $r$ with exactly $1 + s_k$ ascent plateaus. In particular, $\mathcal{Q}\,(P_1 \bigcup N_0) = \{11\}$, $\mathcal{Q}\,(P_1 \bigcup N_1) = \{2211, 1221\}$ and $\mathcal{Q}\,(P_2 \bigcup N_0) = \{1122\}$. Let us first give a definition of labeled Stirling permutations.

**Definition 8.** *Let $\sigma \in \mathcal{Q}\,(P_{1+s_k} \bigcup N_{t_k})$. If $i_1 < i_2 < \ldots < i_{1+s_k}$ are the ascent plateaus of $\sigma$, then we put the superscript labels $2\ell - 1$ before $i_\ell$ and $2\ell$ after it, where $1 \leq \ell \leq 1 + s_k$. In the remaining positions, we put the superscript labels $-1, -2, \ldots, -2t_k$ and $\star$ from left to right.*

For example, the labels of 13324421 are given as follows:

$$^{-1}1^13^23^{-2}2^34^44^{-3}2^{-4}1^\star.$$

Given a MY-sequence $Y_n = (y_1, y_2, \ldots, y_n)$. Now we present a labeled Stirling permutations algorithm (LSP algorithm for short):

($S_1$) Since $\mathcal{Q}\left(P_1 \bigcup N_0\right) = \{11\}$, we take 11 as the start point, which corresponds to $y_1 = \star$. Since 11 can be labeled as $^1 1^2 1^\star$, we distinguish three cases: if $y_2 = 1$, then we get 2211 by inserting 22 to the position with superscript label 1; if $y_2 = 2$, then we get 1221 by inserting 22 to the position with superscript label 2; if $y_2 = \star$, then we we get 1122 by inserting 22 to the position with superscript label $\star$.

($S_2$) When $y_2 = 1$, we take $^1 2^2 2^{-1} 1^{-2} 1^\star$ as the start point. Since $y_3 \in P_1 \bigcup N_1$, we distinguish five cases: if $y_3 = 1$, then we get **33**2211 by inserting 33 to the position with superscript label 1; if $y_3 = 2$, then we get 2**33**211 by inserting 33 to the position with superscript label 2; if $y_3 = -1$, then we get 22**33**11 by inserting 33 to the position with superscript label -1; if $y_3 = -2$, then we get 221**33**1 by inserting 33 to the position with superscript label -2; if $y_3 = \star$, then we get 2211**33** by inserting 33 to the position with superscript label $\star$.

($S_3$) When $y_2 = 2$, we take $^{-1} 1^1 2^2 2^{-2} 1^\star$ as the start point. Note that $y_3 \in P_1 \bigcup N_1$. Along the same lines as the five cases of ($S_2$), we get $133221, 123321, 331221, 122331, 122133$.

($S_4$) When $y_2 = \star$, we take $^1 1^2 1^3 2^4 2^\star$ as the start point. Note that $y_3 \in P_2 \bigcup N_1$. Similarly, we get $331122, 133122, 113322112332, 112233$.

($S_5$) Repeat the above procedure, we get all labeled Stirling permutations.

It is straightforward to show that each such labeled Stirling permutation will be obtained exactly once. Indeed, given a labeled Stirling permutation of order $n$ with $i$ ascent plateaus, we can just read the indices of ascent plateaus. Deleting the pair $(2n)(2n)$ gives the entry $y_n$. Along the same lines, we can get a unique MY-sequence $Y_n$ with $\mathrm{neg}\,(Y_n) + \mathrm{star}\,(Y_n) = i$. In conclusion, LSP algorithm gives the desired proof of the right equality of (10).

**Example 9.** *Let $Y_6 = (\star, 2, -1, 3, -2, 2)$ and let $\sigma = 366314455221$. The correspondence between $Y_6$ and $\sigma$ is built up as follows:*

$$2 \to^1 1^2 1^\star \Leftrightarrow 1221;$$
$$-1 \to^{\text{-1}} 1^1 2^2 2^{-2} 1^\star \Leftrightarrow 331221;$$
$$3 \to^1 3^2 3^{-1} 1^3 2^4 2^{-2} 1^\star \Leftrightarrow 33144221;$$
$$-2 \to^1 3^2 3^{-1} 1^3 4^4 4^{\text{-2}} 2^{-3} 2^{-4} 1^\star \Leftrightarrow 3314455221;$$
$$2 \to^1 3^2 3^{-1} 1^3 4^4 4^5 5^6 5^{-2} 2^{-3} 2^{-4} 1^\star \Leftrightarrow 366314455221.$$

## 5. Proof of (11)

Let us first give a definition of labeled permutations.

**Definition 10.** *Let $\pi \in \mathfrak{S}_n$ with $p$ excedances. If $i_1 < i_2 < \cdots < i_p$ are the excedances, then we put superscript labels $-k$ between $i_k$ and $\pi(i_k)$, where $1 \le k \le p$. In the remaining positions except the first position, we put the superscript labels $1, 2, \ldots, n-p$ from left to right.*

In the following discussion, we shall attach superscript labels to all permutations in $\mathfrak{S}_n$. For example, the labels of $(\mathbf{1}, \mathbf{3}, 7, 4, 2)(5)(\mathbf{6}, 10, 8)(\mathbf{9}, 11)$ are given as follows:

$$(\mathbf{1}\ ^{-1}\ \mathbf{3}^{-2}\ 7\ ^1\ 4\ ^2\ 2\ ^3)(5^4)(\mathbf{6}\ ^{-3}\ 10\ ^5\ 8\ ^6)(\mathbf{9}\ ^{-4}\ 11\ ^7).$$

A labeled permutations algorithm (LP algorithm for short) for generating MY-sequences is given as follows:

Take $(1^1)$ as the start point, which corresponds to the first term $\star$ of a MY-sequence. Let $\pi$ be a labeled permutation in $\mathfrak{S}_n$. There are $n+1$ permutations of $\mathfrak{S}_{n+1}$ can be obtained from $\pi$ by inserting $n+1$ to the positions with superscript labels or as a new cycle $(n+1)$. We distinguish three cases:

- $(C_1)$ If we insert the entry $n+1$ to the position with superscript label $k$, then the $n$-th term of the corresponding MY-sequence is $y_n \in \{2k-1, 2k\}$.
- $(C_2)$ If we insert the entry $n+1$ to the position with superscript label $-\ell$, then the $n$-th term of the corresponding MY-sequence is $y_n \in \{-2\ell+1, -2\ell\}$.
- $(C_3)$ If we insert the entry $n+1$ at the end of $\pi$ to form a new cycle $(n+1)$, then the $n$-th term of the corresponding MY-sequence is $\star$.

In particular, consider $\pi = (1^1)$. Note that $(1,2)$ is obtained from $(1^1)$ by inserting 2 to the position with superscript label 1. Hence $y_2 \in \{1,2\}$ is the MY-sequence that corresponds to $(1,2)$. Note that $(1)(2)$ is obtained from $(1^1)$ by inserting 2 as a new cycle $(2)$. Hence $y_2 = \star$ is the MY-sequence that corresponds to $(1)(2)$. Take $(1^1)$ as a start point. Repeat the LP algorithm $n$ times, it is easy to verify that each MY-sequence of length $n$ will be obtained exactly once. Note that $2^{\mathrm{cdes}\,(\pi)+\mathrm{exc}\,\pi}$ is the number of MY-sequences that corresponds to $\pi$. Therefore, the LP algorithm gives a combinatorial proof of (11).

**Example 11.** *Given $\pi = (1,3,5,2,6)(4)$. The* LP *algorithm can be done if you proceed as follows:*

$$(1^1) \to (1,2) \Leftrightarrow y_2 \in \{1,2\};$$
$$(1\ ^{-1}\ 2\ ^1) \to (1,3,2) \Leftrightarrow y_3 \in \{-1,-2\};$$
$$(1\ ^{-1}\ 3\ ^1\ 2\ ^2) \to (1,3,2)(4) \Leftrightarrow y_4 \in \{\star\};$$
$$(1\ ^{-1}\ 3\ ^1\ 2\ ^2)(4\ ^3) \to (1,3,5,2)(4) \Leftrightarrow y_5 \in \{1,2\};$$
$$(1\ ^{-1}\ 3\ ^{-2}\ 5\ ^1\ 2^2)(4^3) \to (1,3,5,2,6)(4) \Leftrightarrow y_6 \in \{3,4\}.$$

*Therefore, the corresponding* MY-*sequences of $\pi$ are given as follows:*

$$(\star, 1, -1, \star, 1, 3); (\star, 1, -1, \star, 1, 4); (\star, 1, -1, \star, 2, 3); (\star, 1, -1, \star, 2, 4);$$

$$(\star, 1, -2, \star, 1, 3); (\star, 1, -2, \star, 1, 4); (\star, 1, -2, \star, 2, 3); (\star, 1, -2, \star, 2, 4);$$

$$(\star, 2, -1, \star, 1, 3); (\star, 2, -1, \star, 1, 4); (\star, 2, -1, \star, 2, 3); (\star, 2, -1, \star, 2, 4);$$

$$(\star, 2, -2, \star, 1, 3); (\star, 2, -2, \star, 1, 4); (\star, 2, -2, \star, 2, 3); (\star, 2, -2, \star, 2, 4).$$

## References

[1] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, *SIAM J. Discrete Math.* 23 (2008/09), 401–406.

[2] F. Brenti, *q*-Eulerian polynomials arising from Coxeter groups, *European J. Combin.* 15 (1994), 417–441.

[3] F. Brenti, A class of *q*-symmetric functions arising from plethysm, *J. Combin. Theory Ser. A,* 91 (2000), 137–170.

[4] K. Dilks, T.K. Petersen, J.R. Stembridge, Affine descents and the Steinberg torus, *Adv. in Appl. Math.* 42 (2009), 423–444.

[5] D. Foata, M. Schützenberger, Théorie Géométrique des Polynômes Euleriens, Lecture Notes in Mathematics, vol. 138, Springer-Verlag, Berlin-New York, 1970.

[6] I. Gessel and R.P. Stanley, Stirling polynomials, *J. Combin. Theory Ser. A,* 24 (1978), 25–33.

[7] Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A,* 64 (1993), 189–215.

[8] J. Haglund, M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, *European J. Combin.* 33 (2012), 477–487.

[9] S. Janson, M. Kuba and A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, *J. Combin. Theory Ser. A,* 118 (2011), 94–114.

[10] L. Lovász, M. Plummer, Matching theory, Annals of Discrete Mathematics 29, North Holland Publishing Co., Amsterdam, 1986.

[11] S.-M. Ma, A family of two-variable derivative polynomials for tangent and secant, *Electron. J. Combin.* 20(1) (2013), #P11.

[12] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, *European J. Combin.* 34 (2013), 1081–1091.

[13] S.-M. Ma, T. Mansour, The $1/k$-Eulerian polynomials and $k$-Stirling permutations, *Discrete Math.* 338 (2015), 1468–1472.

[14] J.B. Remmel, A.T. Wilson, Block patterns in Stirling permutations, `arXiv:1402.3358`.

[15] C.D. Savage and M.J. Schuster, Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences, *J. Combin. Theory Ser. A,* 119 (2012), 850–870.

[16] C.D. Savage and G. Viswanathan, The $1/k$-Eulerian polynomials, *Electron. J. Combin.* 19 (2012), #P9.

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P.R. China

*E-mail address*: `shimeimapapers@gmail.com` (S.-M. Ma)

Institute of Mathematics, Academia Sinica, Taipei, Taiwan

*E-mail address*: `mayeh@math.sinica.edu.tw` (Y.-N. Yeh)